I ~ TUTORATO ~

TOPOLOGIA IN $\mathbb{R}^m$

① $P = (2, -1) \in \mathbb{R}^2$

$$X = \{ Q = (x,y) : \langle v_{OP}, v_{OQ} \rangle \geq 1 \} \subseteq \mathbb{R}^2$$

$$v_{OP} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad v_{OQ} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \langle v_{OP}, v_{OQ} \rangle = 2x - y$$

$$X = \{ (x,y) \in \mathbb{R}^2 : |2x - y| \geq 1 \}$$

$$|2x - y| \geq 1 \iff 2x - y \geq 1 \cup 2x - y \leq -1$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

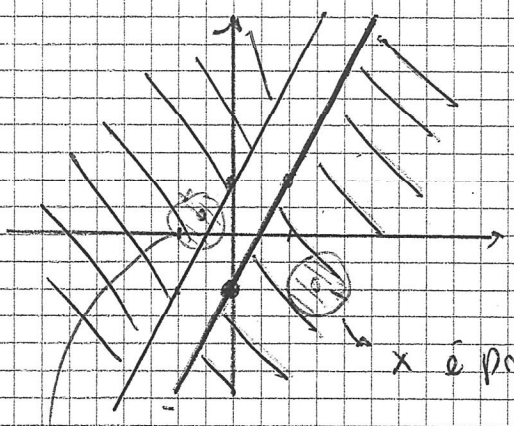
$$y \leq 2x - 1 \cup y \geq 2x + 1$$

X è

x	y
0	-1
1	1

x	y
0	1
-1	-1

UNIONE di due semipiani chiusi



x è pnto interno di X

x è pnto di frontiera per X

X NON è APERTO $\Leftarrow$ ci sono pnti di frontiera

X^c APERTO $\Rightarrow$ X CHIUSO

X non limitato $\Rightarrow$ X non compatto

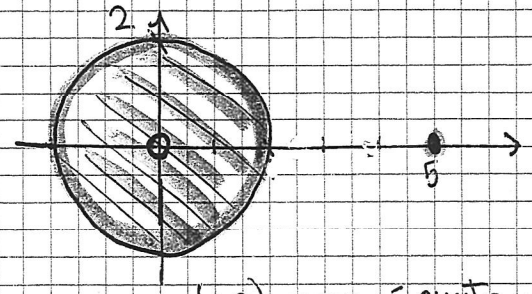
$$\mathring{X} = \{ (x,y) \in \mathbb{R}^2 : |2x - y| > 1 \}$$

$$\overline{X} = X \cup \partial X = X \text{ poiché } X \text{ è chiuso (coincide con la sua chiusura)}$$

$$\partial X = \{ (x,y) \in \mathbb{R}^2 : |2x - y| = 1 \}$$

$$Acc(X) = X$$

② $X = \{(x,y) \in \mathbb{R}^2 : 0 < x^2+y^2 \le 4\} \cup \{(5,0)\}$



$(5,0)$ non è un punto int di X

$\Downarrow$

X non è aperto

$(0,0)$ non è punto interno di $X^c \Rightarrow X^c$ non è aperto

$\Rightarrow X$ non è chiuso

$B_6((0,0)) \supseteq X \Rightarrow X$ è limitato ma non compatto

$\mathring{X} = \{(x,y) \in \mathbb{R}^2 : 0 < x^2+y^2 < 4\} = B_2((0,0)) \setminus \{(0,0)\}$

$\bar{X} = X \cup \{(0,0)\} = \overline{B_2((0,0))} \cup \{(5,0)\}$

$\partial X = \{x^2+y^2 = 4\} \cup \{(0,0),(5,0)\}$

$Acc(X) = \bar{X} \setminus \{(5,0)\} = \overline{B_2((0,0))}$

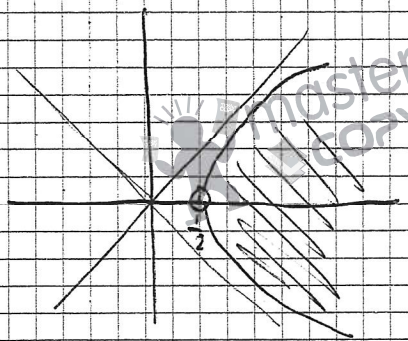
③ $P = (1,0)$, $Q = (-1,0)$

$X = \{R = (x,y) : d(R,Q) - d(R,P) \ge 1, \quad x > \frac{1}{2}\}$

$\sqrt{(x+1)^2+y^2} - \sqrt{(x-1)^2+y^2} \ge 1 \iff$

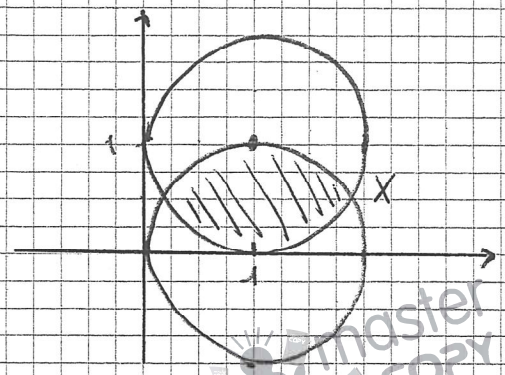
$\iff \begin{cases} 4x^2 - \frac{4}{3}y^2 \ge 1 \\ x \ge \frac{1}{4} \end{cases}$

Iperbole



④

$X = B_2((1,0)) \cap B_1((1,1)) \subseteq \mathbb{R}^2$

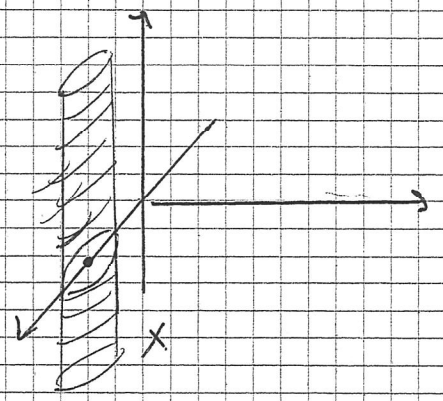


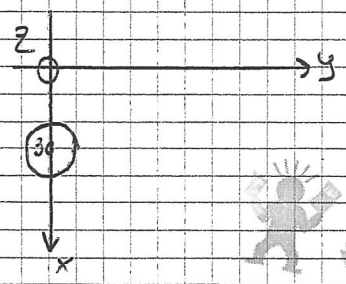
X aperto $\Rightarrow \mathring{X} = X$

$\bar{X} = \overline{B_1((1,0))} \cap \overline{B_1((1,1))}$

$Acc(X) = \bar{X}$

⑤ $X = \{(x,y,z) \in \mathbb{R}^3 : (x-3)^2 + y^2 = 1\}$





$\forall k \in \mathbb{R}$ $X \cap \{z = k\}$ circonf. di raggio 1 e centro $(3,0,k)$

$\Downarrow$

X sup. cilindrica

$\mathring{X} = \emptyset$; $\bar{X} = X$, $\partial X = X$ $Acc(X) = \bar{X} = X$

LIMITI

① $\lim_{(x,y)\to(0,0)} \frac{x-y}{(x+y)^3}$

$X = \text{Dom}(f) = \{(x,y) \in \mathbb{R}^2 : x+y \neq 0\}$

$(0,0) \in Acc(X)$

Teorema Ponte $A \subset X$ t.c. $(0,0) \in Acc(A)$

asse y $\{x=0\} = A$

$\lim_{\substack{(x,y)\to(0,0) \\ x=0}} \frac{x-y}{(x+y)^3} = \lim_{y\to 0} \frac{-y}{y^3} = -\frac{1}{y^2} \to -\infty$

asse x $\{y=0\} = A$

$\lim_{\substack{(x,y)\to(0,0) \\ y=0}} \frac{x-y}{(x+y)^3} = \lim_{x\to 0} \frac{x}{x^3} = \frac{1}{x^2} \to +\infty$

il limite non esiste

② $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+2y^2)}{\sqrt{x^2+y^2}}$

$X = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \neq 0\} = \{\mathbb{R}^2 \setminus \{(0,0)\}\}$

$(0,0) \in Acc(X)$ ha senso il limite

$$\frac{\sin(x^2+2y^2)}{\sqrt{x^2+y^2}} \cdot \frac{x^2+2y^2}{x^2+2y^2}$$

$t = x^2+2y^2 \quad \frac{\sin t}{t} \to 1$ per $t \to 0$

$$f(x,y) = \frac{x^2+2y^2}{\sqrt{x^2+y^2}}$$

asse y $\{x=0\} = A$ $(0,0) \in Acc(A)$

$\lim_{\substack{y\to 0 \\ x=0}} \frac{2y^2}{\sqrt{y^2}} = \frac{2y^2}{|y|} \overset{y\neq 0}{=} 2|y| \to 0$

asse x $\{y=0\} = B$ $(0,0) \in Acc(B)$

$$\lim_{\substack{x\to 0\\ y=0}} \frac{x^2}{\sqrt{x^2}} = \lim_{x\to 0} \frac{x^2}{|x|} \overset{x\to 0}{\leq} |x| \to 0$$

$A = \{y = \lambda x\}$, $\lambda \in \mathbb{R}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=\lambda x}} g(x,y) = \lim_{x\to 0} \frac{x^2+2\lambda^2x^2}{\sqrt{x^2+\lambda^2x^2}} = \lim_{x\to 0} \frac{x^2(1+2\lambda^2)}{\sqrt{x^2(1+\lambda^2)}} \overset{x\neq 0}{=}$$

$$= \lim_{x\to 0} \frac{x^2(1+2\lambda^2)}{|x|\sqrt{1+\lambda^2}} = \lim_{x\to 0} |x| \frac{1+2\lambda^2}{\sqrt{1+\lambda^2}} \to 0 \quad \forall \lambda \in \mathbb{R}$$

$$0 \leq \left| \frac{x^2+2y^2}{\sqrt{x^2+y^2}} - 0 \right| \leq g(x,y)$$

$$0 \leq \frac{x^2+2y^2}{\sqrt{x^2+y^2}} \leq \cdot \frac{2(x^2+y^2)}{\sqrt{x^2+y^2}} = 2\sqrt{x^2+y^2} \to 0$$

per il th del confronto $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+2y^2)}{\sqrt{x^2+y^2}} = 0$

③ $\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^4+y^4}$

$X = \{(x,y) \in \mathbb{R}^2 : x^4+y^4 \neq 0\} \to X = \mathbb{R}^2 \setminus \{(0,0)\}$

$(0,0) \in Acc(X)$

$A = \{y = x^\alpha\}$, $\alpha \geq 0$, $x>0$, $y>0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0^+} \frac{x^3 x^{3\alpha}}{x^4+x^{4\alpha}}$$

$y = x^\alpha$, $\alpha>0$, $x>0$

$$= \lim_{x\to 0^+} \frac{x^{3+3\alpha}}{x^4+x^{4\alpha}} = \lim_{x\to 0^+} \frac{x^{3+3\alpha}}{x^4+o(x^{4\alpha})}$$

trovare un α in modo che non sia $=0$ e lim. rendere esp dem in ordine minimo num e confronto

$4 > 3+3\alpha$

$1 > 3\alpha$ $\quad \alpha < \frac{1}{3}$

$$\lim_{y\to 0} \frac{0}{y^4} \quad y=\lambda x \quad \frac{x^3\lambda^3x^3}{x^4+\lambda^4x^4} = \frac{x^6\lambda^3}{x^4(1+\lambda^4)} \to 0$$