MASTER COPY
Tel. 050 8312126
Cell. 388 9837745

University of Pisa

Aerospace Systems Analysis
Course Code: 663II

Written Exam
Date: January 8th 2016

Second Cycle Degree Programme in Aerospace Engineering
Corso di Laurea Magistrale in Ingegneria Aerospaziale

Student: ***Student Id. N.:***

Answers to questions must be reported on the worksheets provided by the teacher (white sheets with the university logo). Graphical calculators are not allowed.

Le risposte alle domande vanno riportate sui fogli di lavoro forniti dal docente (fogli bianchi con il logo dell'università) Non sono ammesse calcolatrici grafiche.

Exercise 1

The dynamics of an aircraft arresting gear system with cable and hydraulic damper (Figure 1) is described by the following model:

$$\begin{cases} m_{AC}\dot{v}_{AC} = T - k_C\varepsilon_C \\ m_{HD}(\dot{v}_{AC} - \ddot{\varepsilon}_C) = k_C\varepsilon_C - k_V A_{HD}\left(\dfrac{v_{AC} - \dot{\varepsilon}_C}{1-u}\right)^2 \end{cases}$$

where m_{AC} and m_{HD} are the aircraft mass and the damper moving mass, A_{HD} is the damper pushing area, k_C is the cable stiffness, k_V is a parameter characterising the pressure-drop/flow-rate relationship of the braking valve, T is the aircraft thrust, u is the signal controlling the braking valve orifice (*system input*), v_{AC} is the aircraft speed (*system output*), ε_C is the cable strain (*system output*). By assuming a constant aircraft thrust ($T = T_0$),

- **1.1** Determine the system equilibrium with zero input to the braking valve ($u(t) = u_{eq} = 0$) **[3 pt]**
- **1.2** Define, with respect to the initial equilibrium conditions defined at point **1.1**, a linearised standard state-space representation of the ODE problem **[7 pt]**
- **1.3** Determine the transfer functions characterising the system response **[6 pt]**

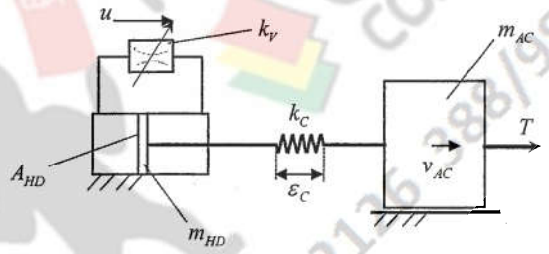


Figure 1 – Aircraft arresting gear system with cable and hydraulic damper.

MASTER COPY
Tel. 050 8312126
Cell. 388 9837745

Exercise 2

The dynamics of a generic system with output y and input u can be represented by the following transfer function:

$$H(s) = -\frac{500s}{(s+0.25)(s^2+8s+250)}$$

Once assumed that a unit-step input is given (i.e. $u(t) = 1(t)$),

- **2.1** explicitly obtain the time response of the system **[5 pt]**
- **2.2** by using the results obtained at point **2.1**, qualitatively plot the time response of the system **[4 pt]**

Exercise 3

Given the closed-loop feedback control represented in the block diagram of Figure 2, define

- **3.1.** the values of the controller gain k that assure the external (or BIBO) stability of the system **[3 pt]**
- **3.2.** the type of instability characterising the response if k reaches the limit values of the stability range defined at point **3.1 [2 pt]**

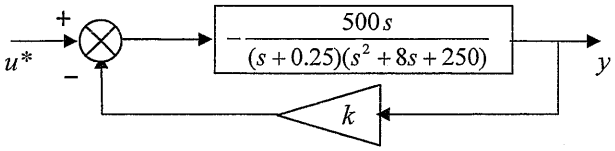


Figure 2 – Feedback control system.

SOLUTION

Numeric results will be considered correct if equal up to the first-digit decimal approximation.

Exercise 1:

Question 1.1

The system dynamics is described by a set of 3rd-order nonlinear time-invariant ODE (2nd order with respect to ε_C and 1st order with respect to v_{AC}), so the dimension of the state vector must equal 3. In particular, let us define

$$\mathbf{x} = [\dot{\varepsilon}_C \quad \varepsilon_C \quad v_{AC}]^T$$

From the definition of static equilibrium condition ($\dot{\mathbf{x}}_{eq} = \mathbf{0}$), we have that

$$\dot{\mathbf{x}}_{eq} = \begin{bmatrix} \ddot{\varepsilon}_{C\,eq} \\ \dot{\varepsilon}_{C\,eq} \\ \dot{v}_{AC\,eq} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \quad \mathbf{x}_{eq} = \begin{bmatrix} 0 \\ \varepsilon_{C\,eq} \\ v_{AC\,eq} \end{bmatrix}$$

where ε_{Ceq} and v_{ACeq} are constant values.
By writing the system equations at an equilibrium in which the aircraft thrust is constant ($T = T_0$) and zero input is fed to the braking valve ($u(t) = u_{eq} = 0$), we have

$$\begin{cases} 0 = T_0 - k_C \varepsilon_{Ceq} \\ 0 = k_C \varepsilon_{Ceq} - k_V A_{HD} \left(\dfrac{v_{ACeq}}{1 - u_{eq}} \right)^2 \end{cases} \Rightarrow \begin{cases} \varepsilon_{Ceq} = \dfrac{T_0}{k_C} \\ v_{AC\,eq} = \sqrt{\dfrac{T_0}{k_V A_{HD}}} \end{cases}$$

MASTER COPY
Tel. 050 8312126
Cell. 388 9837745

Question 1.2

The ODE problem describing the system dynamics can be re-written as follows

$$\mathbf{f}(\dot{\mathbf{x}}, \mathbf{x}, u) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} m_{AC}\dot{x}_3 - T_0 + k_C x_2 \\ \dot{x}_2 - x_1 \\ m_{HD}(\dot{x}_3 - \dot{x}_1) - k_C x_2 + k_V A_{HD} \left(\dfrac{x_3 - x_1}{1 - u} \right)^2 \end{bmatrix}$$

The system linearization is obtained with the first-order approximation of these equations, by assuming that initial conditions are referred to the equilibrium, i.e. $\mathbf{x}(0) = \mathbf{x}_0 = \mathbf{x}_{eq}$; $u(0) = u_{eq}$, and by calculating the following Jacobian matrices:

$$\mathrm{E} = -\left. \frac{\partial \mathbf{f}}{\partial \dot{\mathbf{x}}} \right|_{\mathbf{x}_{eq}, u_{eq}} = -\left[\begin{matrix} \dfrac{\partial f_1}{\partial \dot{x}_1} & \dfrac{\partial f_1}{\partial \dot{x}_2} & \dfrac{\partial f_1}{\partial \dot{x}_3} \\ \dfrac{\partial f_2}{\partial \dot{x}_1} & \dfrac{\partial f_2}{\partial \dot{x}_2} & \dfrac{\partial f_2}{\partial \dot{x}_3} \\ \dfrac{\partial f_3}{\partial \dot{x}_1} & \dfrac{\partial f_3}{\partial \dot{x}_2} & \dfrac{\partial f_3}{\partial \dot{x}_3} \end{matrix} \right]_{\mathbf{x}_{eq}, u_{eq}} = \begin{bmatrix} 0 & 0 & -m_{AC} \\ 0 & -1 & 0 \\ m_{HD} & 0 & -m_{HD} \end{bmatrix}$$

$$\hat{\mathrm{A}} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_{eq},u_{eq}} = \left[\begin{array}{ccc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{array}\right]_{\mathbf{x}_{eq},u_{eq}} = \left[\begin{array}{ccc} 0 & k_C & 0 \\ -1 & 0 & 0 \\ -2k_V A_{HD} v_{AC\,eq} & -k_C & 2k_V A_{HD} v_{AC\,eq} \end{array}\right]$$

$$\hat{\mathrm{B}} = \left.\frac{\partial \mathbf{f}}{\partial u}\right|_{\mathbf{x}_{eq},u_{eq}} = \left[\begin{array}{c} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \end{array}\right]_{\mathbf{x}_{eq},u_{eq}} = \left[\begin{array}{c} 0 \\ 0 \\ 2k_V A_{HD} v_{AC\,eq}^2 \end{array}\right]$$

So we have

$$\mathbf{f} \approx -\mathrm{E}\,\dot{\mathbf{x}} + \hat{\mathrm{A}}(\mathbf{x}-\mathbf{x}_{eq}) + \hat{\mathrm{B}}(u-u_{eq}) \quad \Rightarrow \quad \dot{\mathbf{x}} = \mathrm{E}^{-1}\hat{\mathrm{A}}(\mathbf{x}-\mathbf{x}_{eq}) + \mathrm{E}^{-1}\hat{\mathrm{B}}(u-u_{eq})$$

where

$$\mathrm{E}^{-1} = \frac{\mathrm{adj(E)}}{\mathrm{det(E)}} = \left[\begin{array}{ccc} -1/m_{AC} & 0 & 1/m_{HD} \\ 0 & -1 & 0 \\ -1/m_{AC} & 0 & 0 \end{array}\right]$$

By using the following substitution of variables

$$\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_{eq} \quad ; \quad \tilde{u} = u - u_{eq} = u$$

and considering that the system output is provided by

$$\mathbf{y} = \left[\begin{array}{c} \varepsilon_C \\ v_{AC} \end{array}\right] \quad \Rightarrow \quad \tilde{\mathbf{y}} = \mathbf{y} - \mathbf{y}_{eq} = \left[\begin{array}{c} \varepsilon_C - \varepsilon_{Ceq} \\ v_{AC} - v_{AC\,eq} \end{array}\right]$$

the state-space form of the problem is given by

$$\begin{cases} \dot{\tilde{\mathbf{x}}} = \mathbf{A}\,\tilde{\mathbf{x}} + \mathbf{B}u \\ \tilde{y} = \mathbf{C}\,\tilde{\mathbf{x}} + \mathbf{D}u \end{cases}$$

where

$$\mathbf{A} = \mathrm{E}^{-1}\hat{\mathrm{A}} = \left[\begin{array}{ccc} -\frac{2k_V A_{HD} v_{AC\,eq}}{m_{HD}} & -\frac{k_C}{m_{HD}^*} & \frac{2k_V A_{HD} v_{AC\,eq}}{m_{HD}} \\ 1 & 0 & 0 \\ 0 & -\frac{k_C}{m_{AC}} & 0 \end{array}\right] \qquad \mathbf{B} = \mathrm{E}^{-1}\hat{\mathrm{B}} = \frac{2k_V A_{HD} v_{AC\,eq}^2}{m_{HD}} \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

MASTER COPY
Tel. 050 8312126
Cell. 388 9837745
master copy COPISTERIA 050/8312126 388/9837745

$$\mathbf{C}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad \mathbf{D}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

with $m_{HD}^{*}=\left(\frac{1}{m_{HD}}+\frac{1}{m_{AC}}\right)^{-1}$

MASTER COPY
Tel. 050 831 2126
Cell. 388 9837745

Question 1.3
The linearised perturbative model of the system thus have zero initial conditions, i.e.

$$\begin{cases}\dot{\tilde{\mathbf{x}}}=\mathbf{A}\tilde{\mathbf{x}}+\mathbf{B}u\\ \tilde{y}=\mathbf{C}\tilde{\mathbf{x}}+\mathbf{D}u\end{cases}$$

with $\tilde{\mathbf{x}}(0)=\tilde{\mathbf{x}}_0=[0\ 0\ 0]^T$

and for these problems the system response is given, in the Laplace domain, by

$$\underline{Y}(s)=\left[\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}\right]U(s)=\mathbf{C}\frac{\mathrm{adj}(s\mathbf{I}-\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})}\mathbf{B}U(s)$$

The system poles are given by $\det(s\mathbf{I}-\mathbf{A})=0$, where

$$\begin{vmatrix} s+\frac{2k_V A_{HD} v_{AC\,eq}}{m_{HD}} & \frac{k_C}{m_{HD}^{*}} & -\frac{2k_V A_{HD} v_{AC\,eq}}{m_{HD}}\\ -1 & s & 0\\ 0 & \frac{k_C}{m_{AC}} & s\end{vmatrix}=s\left[s\left(s+\frac{2k_V A_{HD} v_{AC\,eq}}{m_{HD}}\right)+\frac{k_C}{m_{HD}^{*}}\right]+\frac{2k_V k_C A_{HD} v_{AC\,eq}}{m_{AC} m_{HD}}=\ldots$$

$$\ldots=s^3+\frac{2k_V A_{HD} v_{AC\,eq}}{m_{HD}}s^2+\frac{k_C}{m_{HD}^{*}}s+\frac{2k_V k_C A_{HD} v_{AC\,eq}}{m_{AC} m_{HD}}$$

Concerning the calculation of the adjoint matrix $\mathrm{adj}(s\mathbf{I}-\mathbf{A})$, it is worth noting that, for obtaining the Laplace transforms of the system outputs $\underline{Y}(s)$, it is not necessary to calculate all its nine components. Actually, if we generally define

$$\mathrm{adj}(s\mathbf{I}-\mathbf{A})=\begin{bmatrix}m_{11} & m_{12} & m_{13}\\ m_{21} & m_{22} & m_{23}\\ m_{31} & m_{32} & m_{33}\end{bmatrix}$$

and we calculate the transfer functions matrix in symbolic terms

$$\underline{Y}(s)=\frac{2k_V A_{HD} v_{AC\,eq}^2}{m_{HD}\det(s\mathbf{I}-\mathbf{A})}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}m_{11} & m_{12} & m_{13}\\ m_{21} & m_{22} & m_{23}\\ m_{31} & m_{32} & m_{33}\end{bmatrix}\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}U(s)=\ldots$$

$$\ldots=\frac{2k_V A_{HD} v_{AC\,eq}^2}{m_{HD}\det(s\mathbf{I}-\mathbf{A})}\begin{bmatrix}m_{21} & m_{22} & m_{23}\\ m_{31} & m_{32} & m_{33}\end{bmatrix}\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}U(s)=\frac{2k_V A_{HD} v_{AC\,eq}^2}{m_{HD}\det(s\mathbf{I}-\mathbf{A})}\begin{bmatrix}m_{21}\\ m_{31}\end{bmatrix}U(s)$$

we see that only two elements of the adjoint matrix have to be calculated:

$$m_{21} = s \quad ; \quad m_{31} = -\frac{k_C}{m_{AC}}$$

Finally, we have

$$\underline{Y}(s) = \begin{bmatrix} H_1(s) \\ H_2(s) \end{bmatrix} U(s) = \begin{bmatrix} \dfrac{2k_V A_{HD} v_{AC\,eq}^2 \, s}{m_{HD} \det(s\mathbf{I} - \mathbf{A})} \\ -\dfrac{2k_C k_V A_{HD} v_{AC\,eq}^2}{m_{AC} m_{HD} \det(s\mathbf{I} - \mathbf{A})} \end{bmatrix} U(s)$$

MASTER COPY
Tel. 050 8312126
Cell. 388 9837745

Exercise 2:

Question 2.1
The system transfer function has one zero at the origin and three poles

- a real pole located at -0.25 rad/s ;
- and two complex conjugated poles with low damping factor ($\zeta = 0.253$) and natural undamped pulsation $\omega_n = 15.8$ rad/s

Since a unit-step is provided the Laplace transform of the system output is

$$Y(s) = H(s)U(s) = \frac{H(s)}{s} = -\frac{500}{(s+0.25)(s^2+8s+250)}$$

By applying the partial fraction decomposition, we have

$$Y(s) = \frac{c_1}{s+0.25} + \frac{c_2 s + c_3}{s^2+8s+250}$$

where

$$c_1 = (s+0.25)Y(s)\big|_{s=-1} = -\frac{500}{0.25^2 - 2 + 250} \approx -2.015$$

Then, by imposing that

$$c_1(s^2+8s+250) + (c_2 s + c_3)(s+0.25) = -500$$

we obtain

$$c_2 \approx 2.015 \quad ; \quad c_3 \approx 15.62$$

The system time response is thus provided by

$$y(t) = -\mathcal{L}^{-1}\left[\frac{2.015}{s+0.25}\right] + \mathcal{L}^{-1}\left[\frac{2.015s + 15.62}{s^2+8s+250}\right] = -2.015\,e^{-0.25t} + e^{-4t}\left(2.015\cos 15.3t + 0.49\sin 15.3t\right)$$