TEOREMA DI RAPPRESENTAZIONE IN BASE.

Fissati $\beta \in \mathbb{N}$, $\beta > 1$

Dato $x \in \mathbb{R}$, $x \neq 0$

$\exists!\ p \in \mathbb{Z}$, $\{d_i\}_{i \geq 1}$: $d_i \neq 0 \Rightarrow d_{i+1} \neq \beta - 1$

$x = \operatorname{sign}(x)\left(\beta^p \sum_{i=1}^{+\infty} d_i \beta^{-i}\right)$

$F(\beta, r, m, M) = \{x \in \mathbb{R} : x = \pm\left(\beta^p \sum_{i=1}^{r} d_i \beta^{-i}\right)$ con $0 \leq d_i \leq \beta - 1$, $d_1 \neq 0$, $p \in \{-m, \dots, M\}\} \cup \{0\}$

$|F| = 2(M+m+1)(\beta-1)\beta^{r-1} + 1$

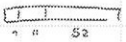
↓ 1 per lo zero, poi ogni numero ha $\beta - 1$ valori

$\beta^{-m-1} \leq |x| \leq \beta^M(1-\beta^{-r})$ → $\omega = \beta^{-m-1}$

$\Omega = \beta^M \sum_{i=1}^{r}(\beta-1)\beta^{-i} = \beta^M(1-\beta^{-r})$

$|y \cdot x| \in \beta^{p+r}$ $\quad x = \beta^p \sum_{i=1}^{r} d_i \beta^{-i}$

$y = \beta^p \sum_{i=1}^{r} d_i \beta^{-i} + \beta^{-r}$



$2^{11} = 2048 \Rightarrow (-1023, 1024)$

$p = \tilde{p} - 1022$

ERRORE RELATIVO DI RAPPRESENTAZIONE.

$\varepsilon_x = \frac{|\tilde{x}-x|}{|x|} \leq u = \beta^{1-r}$

$\frac{|\tilde{x}-x|}{|x|} \leq \frac{|\tilde{x}-\tilde{y}|}{|x|} \leq \frac{\beta^{p+r}}{\beta^{p-1}} = \beta^{1-r} = u$ ⟶ TRONCAMENTO: $u = 2^{-52}$

$\omega = \beta^{p-1}$ = più piccolo numero rappresentabile.

ARROTONDAMENTO: $\frac{1}{2}\beta^{1-r} = 2^{-53}$

$\simeq 1.0^{-16}$

ERRORE TOTALE:

$\epsilon_{TOT} = \epsilon_{Alg} + \epsilon_m$

$\epsilon_{Tot} = \frac{q(\tilde{x}) - p(x)}{q(x)} = \frac{q(\tilde{x}) - p(\tilde{x}) + p(\tilde{x}) - p(x)}{p(x)}$

$= \frac{q(\tilde{x}) - p(\tilde{x})}{q(x)} + \epsilon_m$

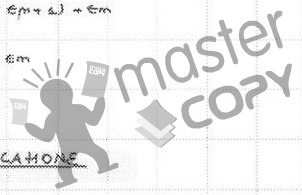
$= \frac{q(\tilde{x}) - p(\tilde{x})}{p(\tilde{x})} \frac{p(\tilde{x})}{q(x)} + \epsilon_m$

$= \epsilon_{Alg}(\epsilon_m + 1) + \epsilon_m$

$\approx C_{Alg} \cdot \epsilon_m$

$\epsilon_m = \frac{p(\tilde{x}) - p(x)}{p(x)} = \frac{p(\tilde{x})}{p(x)} - 1$

$\frac{p(\tilde{x})}{p(x)} = \epsilon_m + 1$



COEFFICIENTE DI AMPLIFICAZIONE

$C_m = \frac{|p'(x)|}{|p(x)|} |x| \leq C_K \cdot u$

$p: \mathbb{R} \to \mathbb{R} \quad p \in C^2(\mathbb{R})$ scelto $x \neq 0$ e $\tilde{x} \neq x$

$p(\tilde{x}) = p(x) + p'(x)(\tilde{x} - x) + p''(\xi)(\tilde{x} - x)^2$ (termine ϵ^2 trascurabile)

$\frac{p(\tilde{x}) - p(x)}{p(x)} = \frac{p'(x)}{p(x)} (\tilde{x} - x) \frac{x}{x}$

$\epsilon_m = C_K \epsilon_K$

PROPRIETÀ:

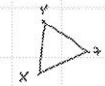
$\mathbb{R}$ $|x - y|$ distanza $\quad \rho(x, y) = |x - y|$

- $|x| \geq 0$ $\quad |x| = 0 \iff x = 0$

  ↓

  distanza è non negativa, e la distanza è nulla $\iff x = y$ (i punti sono uguali)

- $|\alpha x| = |\alpha| \cdot |x|$ (da cui deriva) $|x - y| = |y - x|$
- $|x + y| \leq |x| + |y|$ $\quad |x - z| = |x - y + y - z| \leq |x - y| + |y - z|$ (distanza x da z $\leq$ distanza x da y + distanza y da z)



PROPRIETÀ TRIANGOLARE

O DISUGUAGLIANZA TRIANGOLARE





NORMA MATRICIALE:

Sia $f: \mathbb{R}^{n\times n} \to \mathbb{R}$ tale che:

1. $f(A) \geq 0$ e $f(A)=0 \iff A=0 \quad \forall A \in \mathbb{R}^{n\times n}$
2. $f(\alpha A) = |\alpha| f(A) \quad \forall A \in \mathbb{R}^{n\times n}, \forall \alpha \in \mathbb{R}$
3. $f(A+B) \leq f(A) + f(B) \quad \forall A, B \in \mathbb{R}^{n\times n}$
4. $f(A \cdot B) \leq f(A) f(B) \quad \forall A, B \in \mathbb{R}^{n\times n}$

è una funzione $f: \mathbb{R}^{n\times n} \to \mathbb{R}$ che soddisfa le proprietà:

1. $f(A) \geq 0$ e $f(A)=0 \iff A=0$
2. $f(\alpha \cdot A) = |\alpha| f(A) \quad \forall \alpha \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}$
3. $f(A+B) \leq f(A)+f(B) \quad \forall A, B \in \mathbb{R}^{n\times n}$
4. $f(A \cdot B) \leq f(A) \cdot f(B) \quad \forall A, B \in \mathbb{R}^{n\times n}$ → matrice quadrata

→ $\|AB\| \leq f(A \cdot B)$

NORMA MATRICIALE INDOTTA DA NORMA VETTORIALE

Definiamo $f: \mathbb{R}^{n\times n} \to \mathbb{R}$

come $f(A) = \|A\| = \max \|Ax\|$ dove $x \in \{x \in \mathbb{R}^n : \|x\| \leq 1\}$

1. scelgo $\|\ \|$
2. $\forall A \in \mathbb{R}^{n\times n} : \|x\| \leq 1\}$ è compatto

$h: S \to \mathbb{R} \quad x \to \|Ax\|$ è continua

$\exists x_M \in S : \max h(x) = h(x_M)$

$\max h(x) = \|A\|$

$\|I\| = \max \|I \cdot x\| = \max \|x\| = 1$ → NORMA MATRICIALE

↓

NORMA VETTORIALE

TEOREMA:

$\forall A \in \mathbb{R}^{n\times n}, \forall \varepsilon \in \mathbb{R}^n \quad \|Id\| \geq 1$

$\|A\sigma\| \leq \|A\|\,\|\sigma\|$

Dim: caso $\sigma = 0 \quad \|A\sigma\| = \|0\| = 0$

$\|A\|\,\|\sigma\| = \|A\| \cdot 0 = 0$

caso $\sigma \neq 0 \quad \|A\sigma\| = \|A \frac{\sigma}{\|\sigma\|}\|\,\|\sigma\| \quad \|\sigma\| = 0 \iff \sigma = 0$

$= \|\sigma\| \cdot \|A \frac{\sigma}{\|\sigma\|}\|$

$\frac{\sigma}{\|\sigma\|} \in S \quad \|A\| = \max \|Ax\|$

$\exists x \in \{x \in \mathbb{R}^n : \|x\| = 1\}$

$\|A \frac{\sigma}{\|\sigma\|}\| = \frac{1}{\|\sigma\|} \|A\sigma\| \leq \|A\|$ → $\|A\sigma\| \leq \|A\|\,\|\sigma\|$

$\|A\| = \max \|Ax\| \quad \|x\| = 1$

$\|A\| = \max \sum_{i=1}^{n} |a_{ij}| \quad \forall j = 1 \ldots n$

$\|A\|_\infty = \max \sum_{j=1}^{n} |a_{ij}| \quad \forall i = 1 \ldots n$

$\|A\|_2 = \sqrt{\rho(A^T A)} = \sqrt{\max |\lambda_i|}$

$f(x) = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x^T x} \quad \|x\|_2$

$\rho(x) = \max_{i=1:n} |x_i| \quad \|x\|_\infty$

$\rho(x) = \sum_{i=1}^{n} |x_i| \quad \|x\|_1$

$\|A\|_\infty = \max_{i=1:n} \sum_{j=1}^{n} |a_{ij}| \quad \forall i \in \{1 \ldots n\}$

$\|A\|_1 = \max_{j=1:n} \sum_{i=1}^{n} |a_{ij}|$

$\|A\|_2 = \sqrt{\rho(A^T A)} = \sqrt{\max |\lambda_i|}$

$Ax = b \quad A\hat{x} = \hat{b}$, $A\tilde{x} = b(1+\epsilon)$

$= b + \delta b$

$\hat{b} = b(1+\epsilon) \quad |\epsilon| \le u$

$\hat{b} = b + \underbrace{\delta b}_{\delta b} e \qquad \delta B = \begin{pmatrix} b_1 \epsilon_1 & & \\ & \ddots & \\ & & b_n \epsilon_n \end{pmatrix}$

$\frac{\|\tilde{x} - x\|}{\|x\|} = \frac{\|A^{-1}(b+\delta b) - A^{-1}b\|}{\|b\|} = \frac{\|A^{-1}\delta b\|}{\|x\|} \le \frac{\|A^{-1}\|\,\|\delta b\|}{\|x\|}$

$Ay = b$

$b \neq 0$

$\|Ax\| = \|b\|$

$\|Bx\| \le \|A\|\|x\|$

$\|b\| \le \|A\|\|x\|$

$\|x\| \ge \frac{\|b\|}{\|A\|}$

$\frac{\|\tilde{x} - x\|}{\|x\|} \le \|A^{-1}\|\|A\|\frac{\|\delta b\|}{\|b\|}$

$\|A\|\|A^{-1}\| \ge \|AA^{-1}\| = \|I\| = 1 \neq 0$

$\frac{\|\tilde{x} - x\|}{\|x\|} \le K(A)\frac{\|\delta b\|}{\|b\|}$

$\|\delta B\|_\infty = \left\| \begin{pmatrix} b_1\epsilon_1 & & \\ & \ddots & \\ & & b_n\epsilon_n \end{pmatrix} \right\|_\infty = |b_j \epsilon_j| \le \|b\|_\infty u$

$\frac{\|\tilde{x} - x\|_\infty}{\|x\|_\infty} \le K(A)\frac{\|\delta b\|_\infty}{\|b\|_\infty} \le K(A)u$

ALGORITMO BACKWARD SOSTITUTION: STABILE ALL'ORDINE TRO.

$a(n) = b_0 \quad \forall n, n$

for $k = n-1, \dots, 1$

$K = n-1$ $\quad K = 0$

$n \cdot n + 1 + 1 \cdot (2)$ $\quad n - n + 1 + 1 = (1)$ $\quad O(n^2)$

$\frac{n(n+1)}{2}$

$S = 0;$

for $j = k+1 : n$ $\quad n - k$

$S = S + a(k,j) * x(j);$

end

$x(k) = (b(k) - S)/a(k,k);$

end

end

$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}$ $\quad i$-i mstr. c mkd.

X porto $n-3$ mstr. $\longrightarrow O(n^2)$

n sottrazioni.

TEOREMA ∃! LU

Se $A \in \mathbb{R}^{n \times n}$ se $\det A(1:k;1:k) \neq 0 \quad \forall k = 1:n$

$\Rightarrow \exists!\ LU = A$.

Metodo = Imporre nulla ai membri a meno diversi.

P.B. $n=1$ $\quad l_{11} u_{11} = a_{11}$

$\Downarrow$

$l_{11} = a_{11} \quad u_{11} = a_{11}$

P.I. $\begin{pmatrix} A_{n-1} & y \\ x^T & a_{nn} \end{pmatrix} = \begin{pmatrix} L_{n-1} & 0 \\ w^T & 1 \end{pmatrix} \begin{pmatrix} U_{n-1} & z \\ 0 & u \end{pmatrix}$

$A_{n-1} \Rightarrow \det A(1:k;1:k) \neq 0 \ \forall k = 1:n-1 \Rightarrow \exists!\ w, z, u : A_{n-1} = L_{n-1} U_{n-1}$

$$\begin{cases} A_{n-1} = L_{n-1} \cdot U_{n-1} \\ L_{n-1} z = y \iff z = L_{n-1}^{-1} y \quad \det L \neq 0 \\ w^T U_{n-1} = x^T \iff w^T = x^T U_{n-1}^{-1} \\ x^T z + u = a_{nn} \iff u = a_{nn} - x^T z \end{cases}$$

$A = LU$

$\det(B) = \det(LU)$

$= \det(L)\det(U)$

$\Downarrow \quad \Downarrow$

$\neq 0 \quad \det A \neq 0$

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