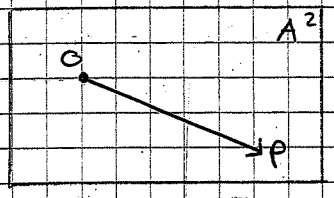
Richiami geometrici

Come si forma uno spazio vettoriale ?



$P \in A^2$

$v = \overrightarrow{OP}$ (oppure si scrive: P − O)

↳ vettore orientato verso P e applicato nel punto di riferimento O

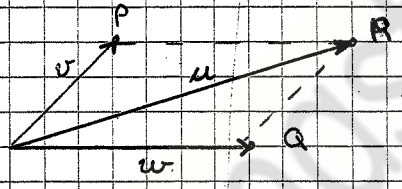
A^1 → retta

A^2 → piano

A^3 → spazio

$\{$ vettori applicati in $O\} = V_O^2 = V^2$

Dati $v, w \in V^2$ trovo $u \in V^2$ t.c : $u = v + w$

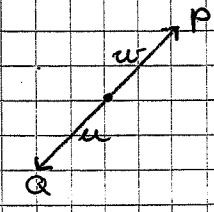


(regola del parallelogramma)

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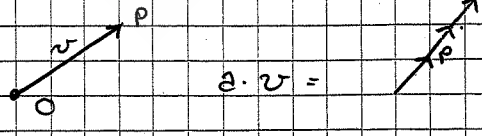
PROPRIETÀ DELLA SOMMA

- commutative : $v + w = w + v$
- associative: $(u + v) + w = u + (v + w)$
- elemento neutro : $\overrightarrow{OO} = O$ $\quad u + O = u$
- opposto : $\forall u \in V_O^2 \ \exists\ w$ t.c. $u + w = w + u = O$



PROPRIETÀ DEL PRODOTTO DI UN VETTORE PER UNO SCALARE

- dato $a \in \mathbb{R}$ $\quad a \cdot v =$



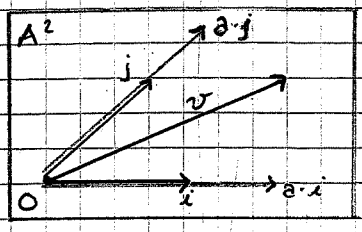
- associative : $(a \cdot b) \cdot v = a(b \cdot v)$
- distributive : $a(v + w) = av + aw$

  $(a + b)v = av + b \cdot v$
- elemento neutro : 1 $\quad 1 \cdot v = v$

  $-1 \cdot v = -v$ (opposto di v)

$(V^2, +)$ è uno spazio vettoriale REALE

ESEMPI:
- $R^d[x] = \{$ polinomi di grado $\leq d$ a coeff. in $\mathbb{R}\}$
- $C^0[a,b] = \{$ funzioni continue su $[a,b]\}$



$v = a \cdot i + bj$

$\forall v \; \exists! \; (a,b)$ t.c $\vec{v} = ai + bj$

NOTAZIONE:
$\exists!$ = esiste unico

$$(\mathbb{R}^2, +) = \left\{ (x,y) \begin{matrix} x \in \mathbb{R} \\ y \in \mathbb{R} \end{matrix} \right\}$$

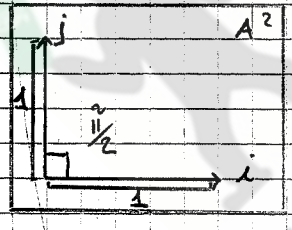
$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

$$a \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} ax_1 \\ ay_1 \end{pmatrix}$$

$V_O^2 \longrightarrow \mathbb{R}^2$ f biettiva (invertibile)

$v = ai + bj \longmapsto (a,b)$

ISOMORFISMO



NOTAZIONE: $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$

è un <u>prodotto scalare</u>

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = x_1 \cdot x_2 + y_1 \cdot y_2$$

$\Downarrow$

identico al prodotto riga × colonna

$$\left[(x_1 \; y_1) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right]$$

PROPRIETÀ DEL PRODOTTO SCALARE

- questo prodotto è BILINEARE

  es: $\langle v+w, u \rangle = \langle v, u \rangle + \langle w, u \rangle \;//\; \langle av, w \rangle = a \langle v, w \rangle$
- SIMMETRICO: $\langle v, u \rangle = \langle u, v \rangle$
- definito positivo

  $v \in \mathbb{R}^2 \quad \langle v, v \rangle \geq 0$

  se $\langle v, v \rangle = 0 \iff v = (0,0)$

NORMA (lunghezza del vettore)

$v \in \mathbb{R}^2 \qquad \|v\| \overset{def}{=} \sqrt{\langle v, v \rangle}$

$v = (x, y) = \sqrt{\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \rangle} = \sqrt{x^2 + y^2}$

$V^2, \langle , \rangle_e$ (euclideo) / $\mathbb{R}^2, \langle , \rangle_c$ (canonico)

$V^2 \longrightarrow \mathbb{R}^2$ applicazione lineare

$v \longmapsto (x_c, y_c)$

$w \longmapsto (x_c, y_c)$

$$\langle v, w \rangle_e \overset{def}{=} \langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \rangle_c$$

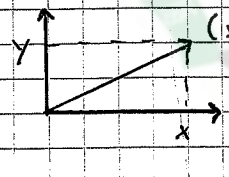
$V^2 \qquad\qquad \mathbb{R}^2$

Praticamente copio sul piano cartesiano una struttura formata da numeri

- Allo stesso modo posso definire la NORMA (algebricamente)

$$\|v\|_e = \sqrt{\langle v, v \rangle_e} = \sqrt{x_1^2 + y_1^2}$$

- La NORMA sul piano cartesiano (geometricamente)



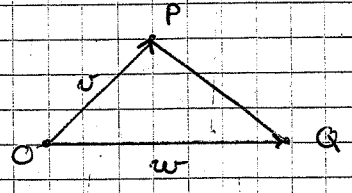
$|v| = \sqrt{x^2 + y^2}$

i 2 procedimenti portano alla stessa conclusione

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Distanza fra 2 punti

$d(v, w) = \|v - w\|$



$d(P, Q) = \|OP - OQ\|$

da un punto di vista geometrico inserendo le coordinate dei punti

$P = (x_1, y_1) \qquad Q = (x_2, y_2)$

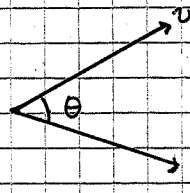
$$d = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$$

Trovare gli angoli

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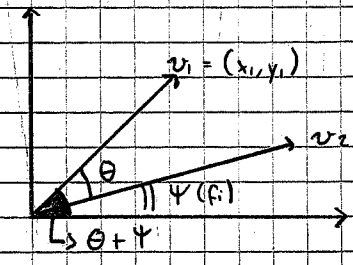
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$(V^2, \langle , \rangle_e)$



$$\cos\theta \overset{def}{=} \frac{\langle v_1, v_2 \rangle}{\|v_1\| \cdot \|v_2\|} \leq |1|$$

Dimostrazione



$$x_1 = \|v_1\| \cdot \cos(\theta + \Psi)$$
$$y_1 = \|v_1\| \cdot \operatorname{sen}(\theta + \Psi)$$
$$x_2 = \|v_2\| \cdot \cos(\Psi)$$
$$y_2 = \|v_2\| \cdot \operatorname{sen}(\Psi)$$

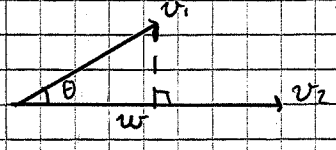
$$\cos(\theta + \Psi) = \cos\theta \cdot \cos\Psi - \sin\theta \cdot \sin\Psi$$
$$\operatorname{sen}(\theta + \Psi) = \sin\theta\cos\psi + \sin\Psi\cos\theta$$

$$\Rightarrow \frac{x_1}{\|v_1\|} = \cos\theta \cdot \cos\Psi - \sin\theta \cdot \sin\Psi \qquad \frac{x_2}{\|v_2\|} = \cos(\Psi)$$
$$\frac{y_1}{\|v_1\|} = \sin\theta \cdot \cos\Psi + \cos\theta \cdot \sin\Psi \qquad \frac{y_2}{\|v_2\|} = \operatorname{sen}(\Psi)$$

sostituisco:

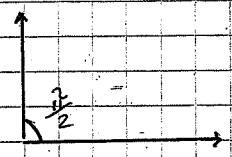
$$\begin{cases} \frac{x_1}{\|v_1\|} = \cos\theta \cdot \frac{x_2}{\|v_2\|} - \sin\theta \cdot \frac{y_2}{\|v_2\|} \\ \frac{y_1}{\|v_1\|} = \sin\theta \frac{x_2}{\|v_2\|} + \cos\theta \frac{y_2}{\|v_2\|} \end{cases}$$

$$\Rightarrow \cos\theta = \frac{\langle v_1, v_2 \rangle}{\|v_1\| \|v_2\|}$$



$$w = \frac{\langle v_1, v_2 \rangle}{\|v_2\|} \cdot \frac{v_2}{\|v_2\|} = \frac{\langle v_1, v_2 \rangle}{\|v_2\|^2} \cdot v_2 = \frac{\langle v_1, v_2 \rangle}{\langle v_2, v_2 \rangle} \cdot v_2$$

Due vettori sono ortogonali quando $\theta = \frac{\pi}{2}$ N.B



$v \perp w \iff \theta = \frac{\pi}{2} \iff \cos\theta = 0 \iff \langle v, w \rangle = 0$

esempio: $v = (2, 1)$

per trovare $w \perp v$ uso il prodotto scalare [?]

$\Rightarrow w = (-1, 2)$

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Spazio metrico (X, d)

Se X è un insieme e se $d: X \cdot X \to \mathbb{R}$

PROPRIETÀ

- POSITIVITÀ: $d(v, w) \geq 0 \quad \forall v, w \in X \quad d(v, w) = 0 \iff v = w$
- SIMMETRIA: $d(v, w) = d(w, v)$
- disuguaglianza triangolare: $d(v, w) \leq d(v, u) + d(u, w)$
  - ↳ in un triangolo ogni lato è minore della somma degli altri 2

esempio:

$(\mathbb{R}, d_b)$ d_b = distanza banale

$$d_b(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$