MASTER COPY
Tel. 050 831 2[illegible]
Cell. 388 983774[illegible]

STATISTICAL SIGNAL PROCESSING

RAPPRESENTAZIONE DI UN SEGNALE TEMPO-DISCRETO

Physical and biological signals can be classified as deterministic and stochastic. Each one is not defined by only one mathematical function. All these signals are continuous. That's a problem because they cannot all fit in a digital memory: we need to ~~digitalize~~ digitalize it in some way.

We get it through, in time or in space. Sampling in space means making a sampling in a specific part of observation space. In sampling ~~also~~ the amplitude is quantized, too.

Considering a periodic signal* $x(t) = x(t - T_0)$, we can say

$$P_S \triangleq \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 \, dt < +\infty \quad \text{(FINITE POWER)}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} S_k \, e^{j 2\pi \frac{k}{T_0} t} \iff S_k = \frac{1}{T_0} \int_{[T_0]} x(t) \, e^{-j 2\pi \frac{k}{T_0} t} \, dt$$

These are synthesis and analysis equation, where S_k are the weights of harmonics in the signal.

Besides, Parseval theorem tells us

$$P_S \triangleq \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 \, dt = \sum_{k=-\infty}^{+\infty} |S_k|^2$$

$$P_{Sx} \triangleq \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \, x^*(t) = \sum_{k=-\infty}^{+\infty} S_k \, X_k^*$$

In this last case, an angle between signals is defined as

$$\theta \triangleq \text{Angle}(x(t), x_c(t))$$

$$\cos\theta \triangleq \frac{P_{Sx}}{\sqrt{P_S \cdot P_x}} \in [-1, 1]$$

It measures the similarity of two signals, one of which is defined as template. If $\cos\theta = 0$, $x(t)$ and $x_c(t)$ are orthogonal. If $\cos\theta = \pm 1$, they are colinear. Furthermore, if a pulse response of an SLS is known, we can calculate

*So, with finite power P_S

MASTER COPY
Tel. 050 831 2126
Cell. 388 9837745

master COPY COPISTERIA
050/8312126 388/9837745
MASTER COPY
Tel. 050 8312126
Cell. 388 9837745

$$y(t) = h(t) \otimes x(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau)\, d\tau \iff y_k = S_k H\left(\frac{k}{T_0}\right)$$

So, if we can save coefficients of discrete forms, we can easily get them in output.
Sampling theorem tells us that, for a band-limited, finite-energy signal

$$x(t) = 2BT_c \sum_{k=-\infty}^{+\infty} x(kT_c)\, \mathrm{sinc}(2B(t-kT_c))$$

So, sampling allows us to have a discrete representation of a continous signal without loss of information, if $f_c \geq 2B$

MASTER COPY
Tel. 050 8312126
Cell. 388 9837745

$$x(t) \iff S(kT_c)$$
$$t \in \mathbb{R} \qquad k \in \mathbb{Z}$$

Generally, basis expansion is the representation of a signal through linear combination of a previously decided function set, $\{\varphi_k(t)\}$.
A signal is said orthonormal when its power is 1 and its cross-power is 0. Basis functions are usually are orthonormals.
Domaage vector is the coefficient vector of a linear combination of basis vector.
In Fourier transform, a problem is vector image's dimension is infinite!
That's why we usually analyse the signal in a short period of time. Now, energy is finite, bandwidth infinite but energy is contained in limited bandwidth
Considering:

$$x(t) = A\, \mathrm{rect}\left(\frac{t - T/2}{T}\right) \iff S(f) = AT\, \mathrm{sinc}(fT) e^{-j\pi fT}$$

It can be demonstrated that, if T and B are chosen in order that $N = 2BT \gg 1$ (N is the number of samples you collect during the observation interval with sampling frequence 2B), for whatever $x(t)$

$$x(t) \overset{2BT_c}{\underset{\Delta}{=}} \sum_{k=0}^{N-1} x\left(\frac{k}{2B}\right) \operatorname{sinc}\left(2B\left(t-\frac{k}{2B}\right)\right) = \sum_{k=0}^{N-1} s_k \varphi_k(t), \quad t \in [0,T)$$

where

$$s_k = \frac{1}{\sqrt{2B}} s\left(\frac{k}{2B}\right) T_c 2B$$

$$\varphi_k(t) = \sqrt{2B} \operatorname{sinc}\left(2B\left(t-\frac{k}{2B}\right)\right), \quad 0 \le k \le N-1 \quad \mathcal{L}_m(\varphi_k(t)) = 1$$

By consequence of that

$$\text{Basis} = \left\{ \varphi_k(t) = \sqrt{2B} \operatorname{sinc}\left(2B\left(t-\frac{k}{2B}\right)\right) \right\}_{k=0}^{N-1}$$

In this case, image vector has finite dimension!

The approximation with sinc is judged by error

$$x_\Delta(t) \overset{\Delta}{=} x(t) - \sum_{k=0}^{N-1} s\left(\frac{k}{2B}\right) \operatorname{sinc}\left(2B\left(t-\frac{k}{2B}\right)\right)$$

If B is large enough, then $E_s \le \alpha E_x$ with $\alpha \le 1$

Sampling is not always the best way to represent a signal; it is only in its bandwidth.

THE DISCRETE KARHUNEN-LOÈVE TRANSFORM

By doing a sampling using the Nyquist theorem, we choose $f_c \ge 2B$ and $2BT >> 1$. Then, we get

$$x(t) = 2BT_c \sum_{k=0}^{N-1} x(kT_c) \operatorname{sinc}(2B(t-kT_c)), \quad t \in [0,T)$$

$$\varphi_k(t) = \left\{ \sqrt{2B} \operatorname{sinc}(2B(t-kT_c)) \right\}_{k=0}^{N-1}$$

So,

$$x(t) \Leftrightarrow \mathbf{x} = \frac{2BT_c}{\sqrt{2B}} \left[x(0) \; x(T_c) \; \dots \; x(T-T_c) \right]^T$$

In this case, the image vector has dimension of N number of samples. In many cases, it can be useful to expand the discrete-time signal obtained by sampling in a set of deterministic discrete-time orthonormal basis functions as follows

$$X[n] = \sum_{i=0}^{K-1} \alpha_i \varphi_i[n]$$

where $K \le N$.

MASTER COPY
Tel. 050 8312126
Cell. 388 9837745

In this case, we need other apriori information, not only the bandwidth. For a finite-energy, time-discrete signal of duration T, the inner product and induced Euclidean norm are defined as

$$(x[n], x_c[n]) \triangleq \sum_{n=0}^{N-1} x[n]\, x_c^*[n] = E_{sx} \quad \text{(mutual energy)}$$

$$\|x[n]\|_2 \triangleq \sqrt{(x[n], x[n])} = \sqrt{\sum_{n=0}^{N-1} |x[n]|^2} = \sqrt{E_s} \quad \text{(energy)}$$

which satisfy our request

coefficients in expansion are

$$\alpha_i = (x_c[n], \varphi_i[n]) = \sum_{n=0}^{N-1} x[n]\, \varphi_i^*[n] = \overline{\Phi}_i^H \underline{x}$$

It is good when these coefficients are uncorrelated. α_i is a random variable, as combination of them. Basis is a deterministic vector.
How to find a basis such as coefficients are uncorrelated? We see that, supposing $K=N$, $E\{x[n]\}=0 \Rightarrow E\{\alpha_i\}$

$$x_c[n] = \sum_{i=1}^{N} \alpha_i \varphi_i[n] \Rightarrow \underline{x} = \sum_{i=1}^{N} \alpha_i \overline{\varphi}_i = \overline{\Phi}\,\underline{\alpha}$$

We can see that

$$\overline{\Phi}^H \underline{x} = \overline{\Phi}^H \overline{\Phi}\,\underline{\alpha} = \underline{\alpha}$$

$\overline{\Phi}$ is such as

$$\overline{\Phi}: \; E\{\underline{\alpha}\,\underline{\alpha}^H\} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N^2 \end{bmatrix} = \Lambda, \quad R_i^2 = E\{|\alpha_i|^2\}$$

$$\overline{\Phi}: \; E\{\underline{\alpha}\,\underline{\alpha}^H\} = E\{\overline{\Phi}^H \underline{x}\,\underline{x}^H \overline{\Phi}\} = \overline{\Phi}^H E\{\underline{x}\,\underline{x}^H\}\overline{\Phi} = \overline{\Phi}^H R_x \overline{\Phi} = \Lambda$$

$$\Lambda = \overline{\Phi}^H R_x \overline{\Phi} \Rightarrow R_x = \overline{\Phi}\,\Lambda\,\overline{\Phi}^H$$

$\overline{\Phi}$ is a matrix of eigenvectors that diagonalizes the covariance matrix. Power of random variables will be their eigenvalues.
This is the Karhunen-Loève expansion and $\underline{\alpha}$ vector is the discrete Karhunen-Loève transform.

MASTER COPY
Tel. 050 8312126
Cell. 388 9837745