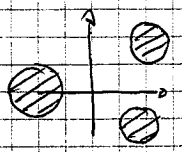
Analisi II

ANALISI 1 1 variabile $f: \Omega \subset \mathbb{R} \to \mathbb{R}$

ANALISI 2 2 o più variabili $f: \Omega \subset \mathbb{R}^n \to \mathbb{R}$

Dominio in Analisi II

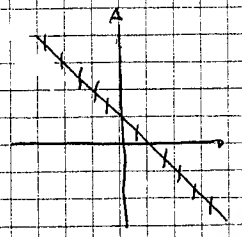
ex Df in $\mathbb{R}^2$



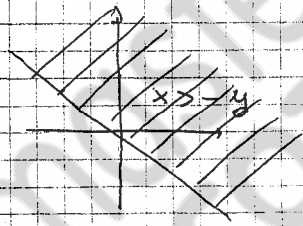
ex $f(x,y) = \dfrac{\sqrt{x^2-y}}{\ln(x+y)}$

$$\begin{cases} x^2-y \geq 0 \\ x+y>0 \\ \ln(x+y) \neq 0 \end{cases} \quad \begin{cases} y \leq x^2 \\ x > -y \\ x+y \neq 1 \end{cases} \quad \begin{cases} y \leq x^2 \\ x > -y \\ x \neq 1-y \end{cases}$$

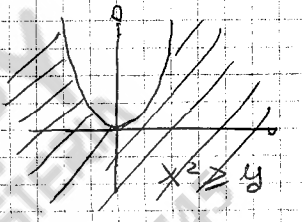
$x+y \neq 1$



$x > -y$

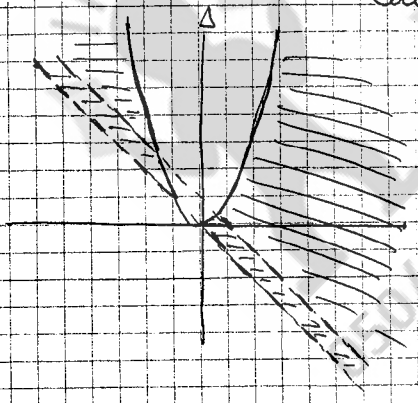


$x^2 \geq y$



(per verificare prendo un punto da una delle due parti)

Df



$= = Df$

$\mathbb{R}^2$ e $\mathbb{R}^n$

$\mathbb{R}^2 = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}\}$ ORIGINE $(0,0)$

$\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R} \ \forall i \ i = 1 \div n\}$ ORIGINE $x_1 = \dots = x_n = 0$

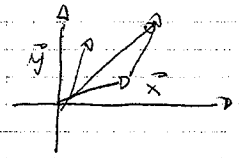
ex $(1, 2, 3, -\sqrt{24}, \frac{1}{2}) \in \mathbb{R}^5$

$(x,y) \in \mathbb{R}^2 \quad (x,y,z) \in \mathbb{R}^3 \quad (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$

SOMMA TRA DUE VETTORI

$\vec{x}(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$

$\vec{y}(y_1, y_2, \dots, y_n) \in \mathbb{R}^n$



$\underbrace{\vec{x} + \vec{y}} = (x_1+y_1, x_2+y_2, \dots, x_n+y_n) \in \mathbb{R}^n$

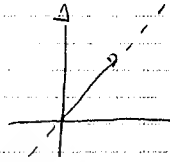
VETTORE SOMMA

PRODOTTO VETTORE – SCALARE

$\vec{x} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$

$\vec{x} = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$

$\lambda\vec{x} = (\lambda x_1, \lambda x_2, \dots, \lambda x_n) \in \mathbb{R}^n$

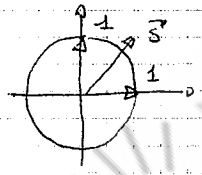


il prodotto scalare tra un vettore e un vettore del vettore

OSSERVAZIONE data f e $D(f) \subset \mathbb{R}^n$ la somma di due vettori non sta sempre nel dominio (quando si verifica questa caratteristica → SPAZIO VETTORIALE)

$\vec{x} \in Df$
$\vec{y} \in Df$ $\not\Rightarrow$ $\vec{x}+\vec{y} \in Df$

ex $f(x,y) = \sqrt{1-x^2-y^2}$ $Df = \{(x,y) : 1-x^2-y^2 \geq 0\}$



se prendo $\vec{x} = (1,0)$ $\vec{y} = (0,1)$

$\vec{x}+\vec{y} = (1,1) \notin Df$

PRODOTTO SCALARE TRA VETTORI

$\vec{x}, \vec{y} \in \mathbb{R}^n$ → $\underbrace{\vec{x} \circ \vec{y}} \in \mathbb{R} = (x_1 \cdot y_1 + x_2 \cdot y_2 + \dots + x_n y_n)$

PRODOTTO SCALARE = un unico numero

Def Dato $\vec{x}(x_1, \dots, x_n)$ $\vec{y} = (y_1, \dots, y_n)$ si definisce

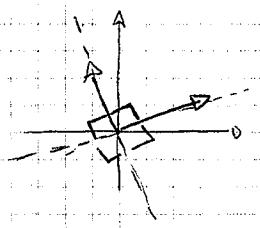
$\vec{x} \circ \vec{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n \in \mathbb{R}$

[x scalar y]

OSSERVAZIONE $\vec{x} \cdot \vec{y} = 0$ quando sono PERPENDICOLARI

$(x_1, x_2) \perp (y_1, y_2)$ ⟷ $x_1 y_1 + x_2 y_2 = 0$

quando i vettori sono rispettivamente $\perp$



NORMA DI UN VETTORE

$\vec{x} = (x_1, x_2, \dots, x_n)$

NORMA: $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \geq 0$ → interpretazione geometrica: DISTANZA DALL'ORIGINE

$\|\vec{x}\| = 0$ ⟷ $\vec{x}$ vettore NULLO

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DISUGUAGLIANZA DI CAUCHY – SCHWARTZ

$\vec{x}, \vec{y} \in \mathbb{R}_n$

$|\vec{x} \circ \vec{y}| \geq 0$

$\boxed{|\vec{x} \circ \vec{y}| \leq \|\vec{x}\| \circ \|\vec{y}\|}$ si può anche scrivere

$$|x_1 y_1 + x_2 y_2 + x_3 y_3 + \dots + x_n y_n| \leq \sqrt{x_1^2 + \dots + x_n^2} \cdot \sqrt{y_1^2 + \dots + y_n^2}$$

$$\rightarrow \left| \sum_{i=1}^{n} x_i y_i \right| \leq \sqrt{\sum_{i=1}^{n} x_i^2} \circ \sqrt{\sum_{i=1}^{n} y_i^2}$$

Dim

OSSERVAZIONE PRELIMINARE

① ⚠ $a, b \in \mathbb{R}$

$$ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$$

Dim

$$(a-b)^2 \geq 0 \rightarrow a^2 + b^2 - 2ab \geq 0 \rightarrow ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$$

CVD

② ⚠ $a, b \in \mathbb{R}$ $\lambda > 0$

$$a \cdot b \leq \frac{1}{2}\lambda a^2 + \frac{1}{2\lambda} b^2$$

$$ab = a\sqrt{\lambda} \cdot \frac{b}{\sqrt{\lambda}} \leq \frac{1}{2}a^2\lambda + \frac{1}{2}b^2 \frac{1}{\lambda}$$

$$ab \leq \frac{1}{2}a^2\lambda + \frac{1}{2\lambda}b^2 \quad \text{CVD}$$

Dim Cauchy – Swcartz

$$\left| \sum_{i=1}^{n} x_i y_i \right| \leq \sum_{i=1}^{n} |x_i y_i| \leq \sum_{i=1}^{n} \left[\frac{1}{2} x_i^2 \lambda + \frac{1}{2\lambda} y_i^2 \right] =$$

$\leq$: proprietà dei valori assoluti $|a+b| \leq |a| + |b|$

$\leq$: oss 2, scelgo $a = |x_i|$, $b = |y_i|$

$$= \sum_{i=1}^{n} \left[\frac{1}{2} x_i^2 \lambda + \frac{1}{2\lambda} y_i^2 \right] = \sum_{i=1}^{n} \frac{1}{2} x_i^2 \lambda + \sum_{i=1}^{n} \frac{1}{2\lambda} y_i^2 =$$

$$= \frac{\lambda}{2}\|\vec{x}\|^2 + \frac{1}{2\lambda}\|\vec{y}\|^2 = \frac{1}{2}\|\vec{x}\|\|\vec{y}\| + \frac{1}{2}\|\vec{x}\|\|\vec{y}\| = \|\vec{x}\|\|\vec{y}\|$$

λ è un parametro libero $\lambda > 0$

$$\lambda = \frac{\|\vec{y}\|}{\|\vec{x}\|}$$

$$\Downarrow$$

$$|\vec{x} \circ \vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$$

CVD

DISTANZA TRA VETTORI IN $\mathbb{R}^n$

$\vec{x}, \vec{y} \in \mathbb{R}^n$

$$d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\| = \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2} = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$$

DISTANZA di $\vec{x}$ da $\vec{y}$

$$\vec{x}-\vec{y} = (x_1-y_1, x_2-y_2, \dots, x_n-y_n)$$

LIMITI IN 2 VARIABILI

- ANALISI 1 $\lim_{x \to x_0} f(x) = l$

$\forall \varepsilon > 0 \ \exists \delta > 0$ tale che $|f(x)-l| < \varepsilon$ se:

1. $|x-x_0| < \delta$ $(I(x_0,\delta))^*$
2. $x \neq x_0$ non conta cosa succede esattamente x_0

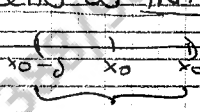
es $\lim_{x \to 0} f(x) = 0$



* $\forall x \in I \cap D \setminus \{x_0\}$

- ANALISI II Generalizzazione del concetto di INTERVALLO

$\{|x-x_0| < \delta\} = I(x_0,\delta)$

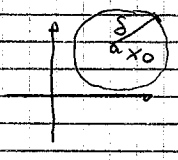


$I(x_0,\delta)$ NON CONTA x_0

diventa

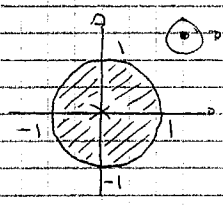
$$\boxed{B(x_0,\delta),\ x_0 \in \mathbb{R}^n}$$

"PALLA" B con centro x_0 e raggio δ



- $f: \underbrace{\Omega \subset \mathbb{R}^n}_{D_f} \to \mathbb{R}$

Per quali p_0 ha senso chiedersi se $\exists \lim_{\vec{x} \to \vec{x}_0} f(\vec{x})$?



- $\lim_{(x,y) \to (2,2)} f(\vec{x})$ NO
- $\lim_{(x,y) \to (0,0)} f(\vec{x})$ SÌ
- $\lim_{(x,y) \to (\frac{1}{2},\frac{1}{2})} f(\vec{x})$ SÌ

HA SENSO PER PUNTI p_0 ADERENTI AL DOMINIO

- **Def** Dato $\Omega \subset \mathbb{R}^n$ si dice $x_0 \in \mathbb{R}^n$ ADERENTE AD Ω se $\forall \delta > 0$ $B(x_0,\delta) \setminus \{x_0\} \cap \Omega \neq \emptyset$

per esempio p_0 isolato (ogni palla bucata D_f senza contare x_0).

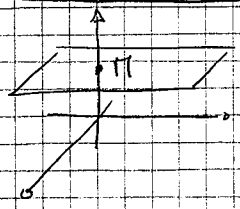
- Quindi data $f: D(f) \to \mathbb{R}$, $x_0 \in \mathbb{R}^n$
  ha senso chiedersi il $\lim_{\vec{x} \to \vec{x_0}} f(\vec{x})$

SE x_0 PTO DI ADERENZA A Df

- $\lim_{\vec{x} \to \vec{x_0}} f(\vec{x}) = \begin{cases} +\infty \\ -\infty \\ \ell \in \mathbb{R} \text{ FINITO} \\ \nexists \text{ (NON ESISTE)} \end{cases}$

$\lim = +\infty$

$$\forall M > 0 \ \exists \delta > 0 : f(x) > M \ \forall x \in B(x_0, \delta) - \{x_0\}$$



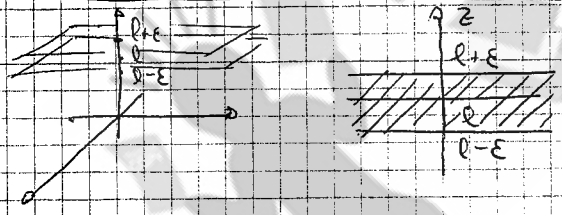
il grafico sta sopra al piano di Π (IPERPIANO M)
$\forall x \in B(x_0, \delta) - \{x_0\}$

$\lim = -\infty$

$$\forall M < 0 \ \exists \delta > 0 : f(x) < M \ \forall x \in B(x_0, \delta) - \{x_0\}$$

$\lim = \ell \in \mathbb{R}$

$$\forall \varepsilon > 0 \ \exists \delta > 0 : |f(x) - \ell| < \varepsilon \ \forall x \in B(x_0, \delta) - \{x_0\}$$



il grafico sta tra i due iperpiani $\ell+\varepsilon$ e $\ell-\varepsilon$
$\forall x \in B(x_0, \delta) - \{x_0\}$

FUNZIONI CONTINUE IN PIÙ VARIABILI

Data $f: \underbrace{D(f)}_{\subseteq \mathbb{R}^n} \to \mathbb{R}$

abbiamo che $x_0 \in Df$ + x_0 PTO DI ADERENZA A Df ⊛

allora f è CONTINUA in x_0 se
$\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}$ e $\ell = f(x_0)$

Def f CONTINUA se $\lim_{x \to x_0} = f(x_0)$ con $x_0 \in Df$ ADERENTE

"TEOREMA" Ogni f_2 ottenuta tramite operazioni algebriche e/o composizioni di f_2 elementari è CONTINUA NEL SUO DOMINIO (no dim.)

⊛ per il limite basta che sia d'aderenza